Decisions, Games, and Negotiation

TODAY’S AGENDA:

Quick review
  – LP, special linear case of Decision making under Certainty
    • Optimization of a single objective
  – Multiple objective decision making under Certainty (PrOACT)
    • Dominance, Even swaps
    • Net Present Value of Cash Flows (special case example)
    • Not necessarily linear tradeoffs
  – Uncertainty
    • Probability Basics, Conditional Probability
    • Bayes’ Rule, Trees, 5-Column method

Today: Uncertainty continued…
  – Introduction to “Random Variables”
    • RVs are a mathematical device used to capture uncertainty.
    • Relies heavily on probability theory
    • Very useful for probabilistic sensitivity analysis and simulation
  – Case: Estimating a Population Proportion
    • Probability Revision using 5-Column method.
Dealing with Uncertainty (continued)

• Unlike a LP, a decision tree model makes uncertainty explicit.

State

T+  .1  8 yrs

.9  T-  10 yrs
Dealing with Uncertainty (continued)

• Unlike a LP, a decision tree model makes uncertainty explicit.

The fact that we don’t know \textit{ex ante} whether \( T^+ \) or \( T^- \) will occur is an uncertainty that the model captures using a probability.
Why do we need to learn about probability distributions?

Note that there are two types of uncertainty here:

- We don’t know whether T+ or T- will occur (i.e. risk)
- We don’t know whether .1 is exactly the right value of the parameter.

We can do a lot of decision analysis (even in the context of uncertainty) without ever talking about a random variable or a probability distribution.
Dealing with Uncertainty (continued)

• The values of parameters in our decision models are often uncertain

A parameter is one of the building blocks of a decision model.

COSTS, BENEFITS, and PROBABILITIES are all parameters.

Parameters are like containers. They contain values.
Estimating Parameter Values

- You may want to gather information about the value of a parameter.

```
Pr( T+ | State) = 0.1

State = 0.9
```

- \( T^+ \): 8 yrs
- \( T^- \): 10 yrs
Case: unknown population proportion

Let $p$ be the unknown population proportion.

Let’s assume this is a parameter in a decision model. Possible examples include:

- Voters who will vote for $X$
- Customers who will buy product $XYZ$
- Students who will pass test
Case: unknown population proportion

• Since \( p \) is uncertain, you can assess your probability distribution for \( p \):
  – Median
  – lower and upper quartiles
  – .01 and .99 extreme points
Case: unknown population proportion

Let $p =$ proportion of left-handed people

Median:

Lower Quartile

Upper Quartile

Lower (.01) extreme

Upper (.99) extreme

Sample: $n =$  
$r =$
Case: unknown population proportion

Now let’s partition the interval from 0 to 1 into 100 equally sized sub-intervals:
  
  from .00 to .01 with central value at .005
  .01 to .02
  .02 to .03
  ...
  .99 1.00

Now let’s make believe that p can take on any one of the 100 values: .005, .015, …, .995.
Case: unknown population proportion

We can think of assessing a prior probability distribution over these 100 possible p-values as the “states”

Now imagine taking a random sample of 25 observations from this population and observing that 10 of these have property XYZ (e.g. left-handedness).

In light of this experimental outcome, how should you revise your probability distribution over these 100 states?
Case: unknown population proportion

<table>
<thead>
<tr>
<th>States</th>
<th>Prior Probs</th>
<th>Likelihood of sample</th>
<th>Product</th>
<th>Posterior Probs</th>
</tr>
</thead>
<tbody>
<tr>
<td>.005</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>.015</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>.325</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>.995</td>
<td>x</td>
<td>x</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>1.00</td>
<td></td>
<td></td>
<td>∑</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Probability of observing sample outcome given p-values
For inferences about an unknown population proportion, the *binomial* distribution plays a central role in objectivist statistics. It is also of some interest to the subjectivist.
Bernoulli Sampling

In a bernoulli trial…

– There are two mutually exclusive and collectively exhaustive outcomes

– The result of each trial is independent of all other trials

– Each trial has the same probability distribution over the two outcomes
Sampling with Replacement

Draw
Observe
Return
Mix

\[ P(r=4 \mid N=10, p=0.3) \]

\[ P(4 \mid 10, 0.3) \]

\begin{align*}
p &= \text{True Proportion of Ss} \\
N &= \text{Sample Size} \\
r &= \text{Number of Ss in Sample}
\end{align*}
Two Drawings with Replacement from a .3-urn

\[ P(S) = 0.3 \times 0.3 = 0.09 \]
\[ P(F) = 0.7 \times 0.3 = 0.21 \]
\[ P(S) = 0.3 \times 0.7 = 0.21 \]
\[ P(F) = 0.7 \times 0.7 = 0.49 \]

Total probability = 0.09 + 0.21 + 0.21 + 0.49 = 1.00

\[ r \quad Probability \]
2 \[ 0.3 \times 0.3 = 0.09 \]
1 \[ 0.3 \times 0.7 = 0.21 \]
1 \[ 0.7 \times 0.3 = 0.21 \]
0 \[ 0.7 \times 0.7 = 0.49 \]
Two Drawings with Replacement from a .3-urn

r   Probability
2   \( .3 \times .3 = .09 \)
1   \( .3 \times .7 = .21 \)
1   \( .7 \times .3 = .21 \)
0   \( .7 \times .7 = .49 \)

Binomial Distribution
N = 2, p = 0.3
### Three Drawings with Replacement from a .7-urn

<table>
<thead>
<tr>
<th>Outcomes</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>FFF</td>
<td>(0.3 \times 0.3 \times 0.3) = 0.027</td>
</tr>
<tr>
<td>FFS, FSF, SFF</td>
<td>(3 \times (0.3 \times 0.3 \times 0.7)) = 0.189</td>
</tr>
<tr>
<td>FSS, SFS, SSF</td>
<td>(3 \times (0.3 \times 0.7 \times 0.7)) = 0.441</td>
</tr>
<tr>
<td>SSS</td>
<td>(0.7 \times 0.7 \times 0.7) = 0.343</td>
</tr>
</tbody>
</table>

**0.7 urn**

- **S** - 7
- **F** - 3
Three Drawings with Replacement from a .7-urn

<table>
<thead>
<tr>
<th>r</th>
<th>Outcomes</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>FFF</td>
<td>.3x.3x.3 = .027</td>
</tr>
<tr>
<td>1</td>
<td>FFS, FSF, SFF</td>
<td>3 x (.3x.3x.7) = .189</td>
</tr>
<tr>
<td>2</td>
<td>FSS, SFS, SSF</td>
<td>3x (.3x.7x.7) = .441</td>
</tr>
<tr>
<td>3</td>
<td>SSS</td>
<td>.7x.7x.7 = .343</td>
</tr>
</tbody>
</table>

0.7 urn

Binomial Distribution: N = 3, p = .7
17 Drawings with Replacement from a 0.325-urn

Let’s Find: \( P( r = 10 \mid n = 17, p = .325) \)

\[
\begin{align*}
\text{SSSSSSSSSSSSSSSSSSSSSSSSSSSSSSSS} & \rightarrow .325^{10} \times .6^{7} \\
\vdots & \vdots \\
\text{SSFFSSSFSSFSFSFS} & \rightarrow .325^{10} \times .6^{7} \\
\vdots & \vdots \\
\text{FFFFFFFFFFSSSSSSSSSSSSSSSS} & \rightarrow .325^{10} \times .6^{7}
\end{align*}
\]

\[
P(r=10\mid n=17, p=0.4) = \frac{C(10,17) \times .325^{10} \times .6^{7}}{\text{Total No. of ways to arrange 10 Ss and 7 Fs}}
\]

\[
C(10, 17) = \frac{17!}{10! \times 7!} = \frac{17 \times 16 \times 15 \times 14 \times 13 \times 12 \times 11}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}
\]
Use of EXCEL to generate the Binomial Distribution

Excel Function:
  Probability Distribution
  \( \Pr(S_n = k|n,p) = \text{BINOMDIST}(k,n,p,0) \)

  Cumulative Distribution
  \( \Pr(S_n \leq k|n,p) = \text{BINOMDIST}(k,n,p,1) \)

Denardo: 8.13
Binomial Distribution in EXCEL

\[ n = 10 \quad \text{p} = .325 \]

<table>
<thead>
<tr>
<th>k</th>
<th>( P(k \mid n, p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.01964</td>
</tr>
<tr>
<td>1</td>
<td>0.09454</td>
</tr>
<tr>
<td>2</td>
<td>0.20484</td>
</tr>
<tr>
<td>3</td>
<td>0.26300</td>
</tr>
<tr>
<td>4</td>
<td>0.22160</td>
</tr>
<tr>
<td>5</td>
<td>0.12804</td>
</tr>
<tr>
<td>6</td>
<td>0.05137</td>
</tr>
<tr>
<td>7</td>
<td>0.01413</td>
</tr>
<tr>
<td>8</td>
<td>0.00255</td>
</tr>
<tr>
<td>9</td>
<td>0.00027</td>
</tr>
<tr>
<td>10</td>
<td>0.00001</td>
</tr>
</tbody>
</table>
Conditional Probability of $k$ given $n = 10$ and $p = 0.325$
# Case: unknown population proportion

We just found: \( P( r = 2 \mid n = 10, p = .325) \)

| States | Prior Probs | Likelihood \( Pr(SR|State) \) | Product Probs | Posterior Probs |
|--------|-------------|-----------------------------|----------------|-----------------|
| .005   | .01         | \( P( r = 2 \mid n = 10, p = .005) \) | .01            | \( = .204 \)    |
| .015   | .01         | \( P( r = 2 \mid n = 10, p = .015) \) | .01            |                 |
| .325   | .01         |                             | .204           |                 |
| .995   | .01         | \( P( r = 2 \mid n = 10, p = .995) \) | .01            |                 |
| 1.00   | \( \sum \)  | 1.00                       |                |                 |

Probability of observing sample outcome given p-values
Take-aways

• There are two types of uncertainty.
  – We know with certainty all the possible future states of the world and we know with certainty the probability that any given state will occur.

  – We know with certainty all the possible future states of the world but we are not sure with what probability each will occur.
Take-aways

• When we don’t know probabilities for sure, we have imperfect information. One way to improve that information is by sampling.

• Today we saw an example of sampling to revise an uncertain estimate of a population proportion.

• Our sampling method was Bernoulli trials. The number of successes in n trials each with \( P(\text{Success}) = p \), was a random variable with a Binomial distribution.