Decisions, Games, and Negotiation

TODAY’S AGENDA:
Limitations of Linear Programming

• Quick review of last lecture
  – Perturbation Theorem
  – Sensitivity Analysis
    • Shadow Price
    • Allowable Increase/Decrease

• Limitations of the LP Technique
  – Proportionality, Additivity, Divisibility, Certainty
Review of Last LP Lecture

• Perturbation Theorem
  – If the data of an LP are perturbed by a small amount, then the optimal solution may change, but its tight constraints stay tight and its slack constraints stay slack [Denardo]
  – Applies to Constraints and Objective Function

• Shadow Price is a breakeven price that is applicable within an allowable range
Linear Programming in Two Dimensions

Geometric Interpretation of Allowable Increases and Decreases, etc.
How to Feed A Chicken

• A major grain supplier decided to produce chicken feed from a mixture of grains and food supplements.
• Each of the possible ingredients had a different price, and each contained different proportions of various nutrients that chickens need each day.
The Problem

Which ingredients, in which proportions, should be combined to meet the nutritional needs of the chickens as inexpensively as possible?
Objectives

- Minimize Costs
- Keep nutrient levels within specified boundaries
Numerical Information

• Corn is priced at 6 cents per pound, Alfalfa at 8 cents per pound.
• Each pound of corn contains 2mg protein, 1mg thiamin, and 14mg fat.
• Each pound of alfalfa contains 1mg protein, 5mg thiamin, and 25mg fat.
More Numerical Information

- Animal nutritionists have determined that chickens require, at a minimum, 15mg of protein per week and 30mg of thiamin.
- It is also known that chickens will not eat more than 285mg of fat per week.
- Given these conditions, how many pounds of corn and how many pounds of alfalfa must be mixed to meet weekly requirements at the lowest possible expense?
Formulating the Objectives Mathematically

• Let x be the number of pounds of corn and y the number of pounds of alfalfa used in the chicken feed.

• Then cost is given by
  \[ c(x,y) = 6x + 8y \]
Formulating the Objectives Mathematically

- Let $x$ be the number of pounds of corn and $y$ the number of pounds of alfalfa used in the chicken feed.
- The protein constraint is given by $2x + y \geq 15$
Formulating the Objectives Mathematically

• Let $x$ be the number of pounds of corn and $y$ the number of pounds of alfalfa used in the chicken feed.

• The thiamin constraint is given by
  
  \[ x + 5y \geq 30 \]
Formulating the Objectives Mathematically

• Let $x$ be the number of pounds of corn and $y$ the number of pounds of alfalfa used in the chicken feed.

• The fat constraint is given by
  
  \[14x + 5y \leq 285\]

• Don’t forget that $x \geq 0$, $y \geq 0$, since we can’t have “negative ingredients.”
Graphing the Problem

• We want to find the *feasible set* of pairs $(x, y)$ which satisfy our constraints
• Since we are constrained by nonnegative $x$ and $y$, we shall be interested in the first quadrant only.
These points all satisfy the protein constraint
These blue points satisfy the thiamin constraint.
These green points satisfy both the protein and thiamin condition.
The yellow points all satisfy the fat constraint
And the pink region is the points which satisfy all constraints!
Solving the Problem

• Now that we have a picture of all the points whose coordinates are possible solutions, solving the problem is a matter of checking all of those points

• To do this, interpret the equation of the objective function $c = 6x + 8y$, as a line in the plane

• Slope-intercept form: $y = -\frac{3x}{4} + \frac{c}{8}$
Solving the Problem

• The y-intercept depends on the value of c, the slope does not

• Thus for different values of c, we get a family of parallel lines.

• These lines represent profiles resulting in the same value of the objective, called isocost lines.
Each blue line represents a zone of equal cost.
Each blue line represents a zone of equal cost.

Notice costs are increasing as we go up and to the right.
The Corner Principle

When a linear function is restricted to a convex polygon, the function assumes its maximum and minimum values at vertices (corners) of the polygon. Therefore, when seeking the extremal values, we need only check a finite number of points!
Based on the diagram, which vertex do you think produces the minimum cost?
<table>
<thead>
<tr>
<th>Vertex</th>
<th>Coordinates</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>(5,5)</td>
<td>70</td>
</tr>
<tr>
<td>Q</td>
<td>(5/2,10)</td>
<td>95</td>
</tr>
<tr>
<td>R</td>
<td>(15,3)</td>
<td>114</td>
</tr>
</tbody>
</table>
Answer

- P is the vertex whose coordinates minimize the objective function
- Therefore, the chicken’s weekly feed should contain 5 pounds of corn and 5 pounds of alfalfa.
- The cost of this mix will be 70 cents.
Recap of the Technique

- Formulate the constraints and the objective function
- Graph the constraints
- Find the convex polygon they determine
- Find the coordinates of the vertices of this polygon
- Evaluate the objective function at each of these vertices
## Spreadsheet Set Up

<table>
<thead>
<tr>
<th>variable</th>
<th>corn</th>
<th>alfafa</th>
<th>total</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>protein</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>&gt;= 16</td>
</tr>
<tr>
<td>thiamin</td>
<td>1</td>
<td>5</td>
<td>6</td>
<td>&gt;= 30</td>
</tr>
<tr>
<td>fat</td>
<td>14</td>
<td>5</td>
<td>19</td>
<td>&lt;= 285</td>
</tr>
<tr>
<td>contribution</td>
<td>6</td>
<td>8</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>var value</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>
## Spreadsheet Solution

<table>
<thead>
<tr>
<th>variable</th>
<th>corn</th>
<th>alfafa</th>
<th>total</th>
<th>rhs</th>
</tr>
</thead>
<tbody>
<tr>
<td>protein</td>
<td>2</td>
<td>1</td>
<td>15</td>
<td>&gt;= 16</td>
</tr>
<tr>
<td>thiamin</td>
<td>1</td>
<td>5</td>
<td>30</td>
<td>&gt;= 30</td>
</tr>
<tr>
<td>fat</td>
<td>14</td>
<td>5</td>
<td>95</td>
<td>&lt;= 285</td>
</tr>
<tr>
<td>contribution</td>
<td>6</td>
<td>8</td>
<td>70</td>
<td></td>
</tr>
<tr>
<td>var value</td>
<td>5</td>
<td>5</td>
<td>50</td>
<td></td>
</tr>
</tbody>
</table>

CHICKEN FEED
# Sensitivity Report

## Adjustable Cells

<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Reduced Cost</th>
<th>Objective Coefficient</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B$9</td>
<td>var value</td>
<td>5</td>
<td>0</td>
<td>6</td>
<td>10</td>
<td>4.4</td>
</tr>
<tr>
<td>$C$9</td>
<td>var value</td>
<td>5</td>
<td>0</td>
<td>8</td>
<td>22</td>
<td>5</td>
</tr>
</tbody>
</table>

## Constraints

<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Shadow Price</th>
<th>Constraint R.H. Side</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D$5</td>
<td>protein total</td>
<td>15</td>
<td>2.4444444444</td>
<td>15</td>
<td>26.30769231</td>
<td>9</td>
</tr>
<tr>
<td>$D$6</td>
<td>thiamin total</td>
<td>30</td>
<td>1.1111111111</td>
<td>30</td>
<td>45</td>
<td>22.5</td>
</tr>
<tr>
<td>$D$7</td>
<td>fat total</td>
<td>95</td>
<td>0</td>
<td>285</td>
<td>1E+30</td>
<td>190</td>
</tr>
</tbody>
</table>
What if the R.H. side of the protein constraint was reduced from 15 to 6?
What if the cost of corn increased from 6 to 16?
Danger!

- This technique cannot be applied to every constrained optimization problem!
- We have made several key assumptions which make the solution so simple
- What are they?
Key Assumptions

• **Linearity**: When either the objective function or the constraints are nonlinear, we may have more points to check, and the vertices may not be extremal after all
Key Assumptions

- Linearity
- **Nonnegativity:** all values of the choice variables are constrained to be positive or zero. This is just an additional set of constraints, but it models the fact that you can’t put a negative amount of an ingredient in your chicken feed!
Key Assumptions

• Linearity
• Nonnegativity
• **Divisibility:** Fractional values of the choice variables are acceptable, not just integers. We know what it means to use 4.5 pounds of corn, etc. Other times, fractional values can be interpreted as “rates,” which is OK for fractions. Integer programming is much harder!
Key Assumptions

• Linearity
• Nonnegativity
• Divisibility
• **Certainty:** All bets are off (pun regretted) unless we know that the constraints and the objective explicitly. However, there is a robustness under small “perturbations” as we will see later
Limitations

Linearity: Proportionality and Additivity

• Proportionality
  – The contribution of each activity to the value of the objective function is proportional to the level of the activity $x_j$, as represented by the function $c_j x_j$ term of the objective function. The same is true for each constraint. (Hillier & Lieberman)

• Additivity
  – Every function in an LP is the sum of individual contributions of the respective activities. (Hillier & Lieberman)
Limitations of LP

• Linearity: Proportionality and Additivity
  – Diseconomy of scale [OK]
    • Diseconomies are indeed nonlinear, however, LP can accommodate decreasing marginal returns (or, similarly, increasing marginal cost).
    • Why? Because you want to first produce product with the higher return before that with lower return.
    • Method: Separate decision variable $S$ into $S_1$ and $S_2$ and create a new functional constraint on $S_1$.

Are economies of scale allowed in LP?
Limitations

• Linearity
  – Economy of scale [Not permissible]
    • Potential Method: Separate decision variable S into S1 and S2 and create a new functional constraint on S1.
    • SOLVER will set S1 to zero and recommend producing just S2.
    • There is no way to tell solver, that S1’s capacity must be exhausted before S2 is produced.
Limitations

Certainty

– The value assigned to each parameter of an LP is assumed to be a known constant (Hillier&Lieberman).
– LP does not explicitly address uncertainty.
– Data is usually not 100% certain, but...
– Sensitivity analysis gives the decision maker an idea where uncertainty is critical.
– “Robustness”
Limitations

Divisibility

– Remember what happened when we relaxed the body shop constraint?

"Relaxed" bodyshop constraint

<table>
<thead>
<tr>
<th>Shop</th>
<th>S</th>
<th>F</th>
<th>L</th>
<th>Contribution</th>
<th>Value of Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Engine</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>120</td>
<td>19.5</td>
</tr>
<tr>
<td>Body</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>81</td>
<td>30.75</td>
</tr>
<tr>
<td>Standard Finishing</td>
<td>2</td>
<td></td>
<td></td>
<td>39</td>
<td>0</td>
</tr>
<tr>
<td>Fancy Finishing</td>
<td>3</td>
<td></td>
<td></td>
<td>92.25</td>
<td>0</td>
</tr>
<tr>
<td>Lux Finishing</td>
<td></td>
<td>2</td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Denardo Table 3.2 p.51

How useful is 0.75 of a Fancy Car?
Limitations

Divisibility

– Decision variables are allowed to have any values, including noninteger values (Hillier&Lieberman).

– To handle indivisibility there is a family of methods for integer programming (IP).

– But often LP works as a relaxation of the constraint of integrality.
When LP doesn’t fit our problem

- Multiple Objectives…