B. Required

3.

In a city, 87% of the households have TVs, 45% have stereos and 38% have both. Let A denote the event in which a randomly-selected household has stereo (audio) and let V denote the event in which a randomly-selected household has visual (TV). For any events A and V,

\[ P(A \cup V) = P(A) + P(V) - P(A \cap V), \]

and we are given \( P(V) = 0.87, \) \( P(A) = 0.45 \) and \( P(A \cap V) = 0.38. \) Thus,

\[ P(A \cup V) = 0.87 + 0.45 - 0.38 = 0.94, \]

so that 94% of the households have at least one of these appliances.

6.

<table>
<thead>
<tr>
<th>Type of Car</th>
<th>Buy - Cash</th>
<th>Buy - Finance</th>
<th>Lease</th>
<th>sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>New</td>
<td>22</td>
<td>15</td>
<td>34</td>
<td>71</td>
</tr>
<tr>
<td>Used</td>
<td>51</td>
<td>18</td>
<td>0</td>
<td>69</td>
</tr>
<tr>
<td>sum</td>
<td>73</td>
<td>33</td>
<td>34</td>
<td>140</td>
</tr>
</tbody>
</table>

(a) Of the last 140 transactions where a car was sold or leased, 71 were new cars and 69 were used, so the probability that the next transaction is a new car is approximately \( \frac{71}{140} = 0.507. \)

(b) The question is not well-worded and full credit will be given for the problem to any student who made an attempt to work on it.

Of the last 106 (= 22 + 15 + 51 + 18) transactions where a car was sold, 15 were new cars and were financed, so the probability that the next will be is approximately \( \frac{15}{106} = 0.142. \)

(c) Of the 37 (= 22 + 15) new cars that were purchased, 15 were financed, and that’s about 40% because \( \frac{15}{37} = 0.405. \) Of the 69 (= 51 + 18) used cars that were purchased, 18 were financed, and that’s about 26% because \( \frac{18}{69} = 0.261. \) It looks as though new car purchases are nearly twice as likely to be financed as are used car purchases.
Alternately, we could look at \( P(\text{car purchased is new}) \) and compare this with \( P(\text{car purchased is new} \mid \text{car purchased with cash}) \) and \( P(\text{car purchased is new} \mid \text{car financed}) \), using the fact that \( P(A \mid B) = P(A) \) if and only if \( A \) and \( B \) are independent.

\[
P(\text{car purchased is new} \mid \text{car purchased with cash}) = \frac{22}{22 + 51} = 0.30
\]
\[
P(\text{car purchased is new} \mid \text{car purchased on finance}) = \frac{15}{15 + 18} = 0.46
\]
\[
P(\text{car purchased is new}) = \frac{22 + 15}{22 + 51 + 15 + 18} = 0.35
\]

We thus conclude that “car purchased is new” is not independent of the type of payment.

C. Choices

1. Two balanced dice are tossed, one white one red.
   (a) Imagine that the red die is tossed first. The event \( w = r \) occurs if the white die is the same number as the red, so
      \[
P(w = r) = \frac{1}{6}, \text{ and } P(w \neq r) = \frac{5}{6}.
\]
   (b) The event \( w \neq r \) has probability \( \frac{5}{6} \), and half of the outcomes in it have \( w < r \), so
      \[
p(w < r) = \frac{5}{12}.
\]
   (c) The probability
      \[
P(w = 3 \mid w < r) = \frac{P(w = 3, w < r)}{P(w < r)} = \frac{\frac{3}{36}}{\frac{5}{12}} = \frac{1}{5}.
\]
   (d) Since \( P(w = 3) = \frac{1}{6} \) and \( P(w = 3 \mid w < r) = \frac{1}{5} \), the events \( w = 3 \) and \( w < r \) are dependent.
   (e) It’s easy to see that
      \[
P(w = 3 \mid w = r) = \frac{P(w = r = 3)}{P(w = r)} = \frac{\frac{1}{36}}{\frac{6}{36}} = \frac{1}{6} = P(w = 3)
\]
So knowing that \( w = r \) provides no information about whether or not \( w = 3 \). Hence, knowing that \( w \neq r \) provides no information about whether or not \( w = 3 \); these two events are independent.
2. With probability $1/3$, the die will come up a $1$ or $2$, in which case the ball will be drawn from urn A, which has 4 white balls and 8 black balls. With probability $2/3$, the die will come up a $2, 4, 5$ or $6$, in which case the ball will be drawn from urn B, which has 9 white balls and 3 black balls.

(a) The tree that appears above describes this experiment.

(b) A white ball is drawn with probability

$$P(W) = (1/3)(1/3) + (2/3)(3/4) = (1/9) + (1/2) = 11/18$$

(c) Given that a white ball is drawn the probability it came from urn A is given by

$$P(A \mid W) = \frac{P(A, W)}{P(W)} = \frac{1/9}{11/18} = 2/11.$$
4. The formal definition that events $A$ and $B$ are independent is that

$P(A) = P(A | B)$.

For events $A$ and $B$ that satisfy $(1)$, we wish to show that $P(B | A) = P(B)$. The definition of conditional probability gives

$$P(B | A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A | B)P(B)}{P(A | B)} = P(B)$$

where the law of conditional probability has been used in the numerator of $(2)$ and where equation $(1)$ has been used in the denominator.

5. The formal definition that events $A$ and $B$ are independent is that

$P(A) = P(A | B)$.

For events $A$ and $B$ that satisfy $(1)$, we wish to show that $P(A | B) = P(A)$. Following the hint, we observe that

$$P(A) = P(A \cap B) + P(A \cap B)$$

and then use (9) twice to get

$$P(A) = P(A | B)P(B) + P(A | B)P(B),$$

and then (1) to get

$$P(A) = P(A)P(B) + P(A | B)P(B),$$

so that

$$P(A) [1 - P(B)] = P(A | B)P(B).$$

But $P(B) = 1 - P(B)$, so dividing the above by $P(B)$ gives

$$P(A) = P(A | B),$$

as desired.
If a deck is shuffled ideally, each of the fifty two cards is equally likely to occur in any position within that deck.

(a) To compute the probability of the particular straight, 6–7–8–9–10, we could draw a probability tree. The:

- First card drawn could be any one of 20 (a 6, 7, 8 9 or 10) of the 52 cards.
- Given that the first card is part of this straight, the second could be any one of 16 (a 6, 7, 8 9 or 10, but not the first) of the remaining 51 cards.
- Given that the first and second are part of the straight, the third could be any of 12 of 50.
- Given the above, the fourth could be any of 8 of 49.
- Given the above, the fifth could be any of 4 of 48.

Thus,

\[
P(\text{draw this straight}) = \frac{20 \times 16 \times 12 \times 8 \times 4}{52 \times 51 \times 50 \times 49 \times 48} = 0.000394
\]

(b) The lowest card in a straight could be A, 2, 3, …, 10, so

\[
P(\text{draw any straight}) = (10) \times P(\text{draw a particular straight}) = 0.00394
\]

13.

Two fair dice are tossed, and the sum S of their pips is counted. There are 6 outcomes in the event S = 7, namely, (6, 1), (5, 2), (4, 3), (3, 4), (2, 5), and (1, 6). Each of these outcomes occurs with probability \(1/6)(1/6) = (1/36)\), so

\[
P(S = 7) = (6)(1/36) = 1/6
\]
(a) Rows 5-7 of the spreadsheet that follows give statistics on those toys that have been tried out on focus groups and produced. Of the 102 toys that have been tried out on the focus group and subsequently produced, 49 have been hits. So a new toy that is tried out on the focus group and is subsequently produced has probability of roughly $49/102 = 0.48$ of being a hit.

(b) If, in addition, the reaction was favorable, the probability that it will be a hit is roughly $30/40 = 0.75$.

(c) For a particular toy, the company has a (prior) probability $P(\text{Hit}) = 0.8$ that it will be a hit before trying it out on the focus group. Given a favorable reaction of the focus group, the probability $P(\text{Hit} | \text{Favorable})$ is (yet another) Bayes calculation, and cell B19 reports that $P(\text{hit} | \text{Favorable}) = 0.928$.

<table>
<thead>
<tr>
<th>focus group's reactions</th>
<th>fav</th>
<th>neutral</th>
<th>neg</th>
<th>any</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hit</td>
<td>30</td>
<td>15</td>
<td>4</td>
<td>49</td>
</tr>
<tr>
<td>Flop</td>
<td>10</td>
<td>15</td>
<td>28</td>
<td>53</td>
</tr>
<tr>
<td>either</td>
<td>40</td>
<td>30</td>
<td>32</td>
<td>102</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>conditional probabilities</th>
<th>fav</th>
<th>neutral</th>
<th>neg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hit</td>
<td>0.612245</td>
<td>0.306122</td>
<td>0.081633</td>
</tr>
<tr>
<td>Flop</td>
<td>0.188679</td>
<td>0.283019</td>
<td>0.528302</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>joint probabilities</th>
<th>fav</th>
<th>neutral</th>
<th>neg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hit</td>
<td>0.489796</td>
<td>0.244898</td>
<td>0.065306</td>
</tr>
<tr>
<td>Flop</td>
<td>0.037736</td>
<td>0.056604</td>
<td>0.10566</td>
</tr>
<tr>
<td></td>
<td>0.527532</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.928467</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
18.

\[
P(\text{not ID}) = 0.98 \\
P(\text{pos} \mid \text{ID}) = 0.9 \\
P(\text{neg} \mid \text{not ID}) = 0.99
\]

<table>
<thead>
<tr>
<th></th>
<th>pos</th>
<th>neg</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>ID</td>
<td>0.018</td>
<td>0.002</td>
<td>0.02</td>
</tr>
<tr>
<td>not ID</td>
<td>0.0098</td>
<td>0.9702</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>0.0278</td>
<td>0.9722</td>
<td></td>
</tr>
</tbody>
</table>

Posterior -- given pos

\[
P(\text{ID} \mid \text{pos}) = 0.647482 \\
P(\text{not ID} \mid \text{pos}) = 0.352518
\]

\[
P(\text{ID} \mid \text{neg}) = 0.002057 \\
P(\text{not ID} \mid \text{neg}) = 0.997943
\]

\[
P(\text{POS} \mid \text{ID}) = 0.95 \\
P(\text{NEG} \mid \text{not ID}) = 0.99
\]

The second table is assuming a positive result on the first test

<table>
<thead>
<tr>
<th></th>
<th>POS</th>
<th>NEG</th>
</tr>
</thead>
<tbody>
<tr>
<td>ID</td>
<td>0.615108</td>
<td>0.032374</td>
</tr>
<tr>
<td>not ID</td>
<td>0.003525</td>
<td>0.348993</td>
</tr>
<tr>
<td></td>
<td>0.618633</td>
<td></td>
</tr>
</tbody>
</table>

Post/posterior -- given pos, POS

\[
P(\text{ID} \mid \text{pos} \& \text{POS}) = 0.994302 \\
P(\text{not ID} \mid \text{pos} \& \text{POS}) = 0.005698
\]

The spreadsheet above processes the test outcomes sequentially. On it, “pos” and “neg” are the outcomes of the first test, while “POS” and “NEG” are the outcomes of the second test.

(a) This spreadsheet computation gives:

\[
P(\text{ID} \mid \text{pos}) = 0.647482 \\
P(\text{ID} \mid \text{pos, POS}) = 0.994302
\]

(b) The more accurate test – the one with outcomes POS and NEG – is likely to come at a higher price. The less accurate test removes from consideration 99.7943% of the athletes who are not ID drug users. Suppose, for instance, the less accurate test costs $10 per administration and that the more accurate test costs $100 per administration. The expected cost of administration is

\[
$10 + (0.0278) ($100) = $12.78 \text{ per athlete},
\]

compared with $100 per athlete if both tests were administered simultaneously.