Part B: Required Problems

1. Consider the following functions:
   (a) $7x + 3y$ linear
   (b) $5x$ linear
   (c) $7x + 4y^2 + 7$ nonlinear
   (d) $x + xy + 13y$ nonlinear

Which ones are linear? Which ones are nonlinear? Write down one more linear function and one more nonlinear function. Which functions ones are valid objective functions for a linear program? Explain your answer.

In the context of linear programming, linear functions satisfy the proportionality and additivity properties. Every additional unit of a particular decision variable contributes the same amount (the value of its coefficient) to the function’s value and the contribution of each decision variable independent of the other decision variables. The functional form that captures both of these properties looks like: $Z = C_1X_1 + C_2X_2 + ... + C_nX_n$. Only linear functions are valid objective functions for a linear program (that is, (a) and (b)).


(A Woodworking Shop) A woodworking shop makes two products, cabinets and tables. Each cabinet sells for $3100, and each table sells for $2100. The raw materials that are needed to produce each cabinet and table cost $1200 and $700, respectively. The variable labor costs needed to make each cabinet and table are $1100 and $900, respectively. The company’s carpentry shop has a capacity of 120 hours per week. Its finishing shop has a capacity of 80 hours per week. Making each cabinet requires 20 hours of carpentry and 15 hours of finishing. Making each table requires 10 hours of carpentry and 10 hours of finishing. The company wishes to determine the product mix that maximizes profit.

(a) Create a table showing all relevant data (use Table 2.1 on page 19 as reference).
(b) Write down a linear program whose optimal solution maximizes profit.
(c) Use Solver to find this optimal solution. Explain why the solution is what it is.
(d) A way has been found to reduce the time needed to finish each cabinet from 15 hours to 12. Reformulate the linear program. Solve it again. Does the optimal solution change? If so, is there anything surprising about the way in which it changes?
(a) Table

<table>
<thead>
<tr>
<th></th>
<th>Cabinets</th>
<th>Tables</th>
</tr>
</thead>
<tbody>
<tr>
<td>+ Price</td>
<td>3100</td>
<td>2100</td>
</tr>
<tr>
<td>- Material</td>
<td>1200</td>
<td>700</td>
</tr>
<tr>
<td>- Labor</td>
<td>1100</td>
<td>900</td>
</tr>
<tr>
<td>= Profit</td>
<td>800</td>
<td>500</td>
</tr>
</tbody>
</table>

(b) Linear Program and (c) Optimal Solution

<table>
<thead>
<tr>
<th></th>
<th>Cabinets</th>
<th>Tables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carpentry</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>Finishing</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>Contribution</td>
<td>800</td>
<td>500</td>
</tr>
<tr>
<td>Value</td>
<td>5.33</td>
<td>0</td>
</tr>
</tbody>
</table>

Constraints

<table>
<thead>
<tr>
<th>Cell</th>
<th>Name</th>
<th>Final Value</th>
<th>Shadow Price</th>
<th>Constraint R.H. Side</th>
<th>Allowable Increase</th>
<th>Allowable Decrease</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$15</td>
<td>Carpentry</td>
<td>106.666667</td>
<td>0</td>
<td>120</td>
<td>1E+30</td>
<td>13.33333333</td>
</tr>
<tr>
<td>$E$16</td>
<td>Finishing</td>
<td>80</td>
<td>53.33333333</td>
<td>80</td>
<td>10</td>
<td>80</td>
</tr>
</tbody>
</table>

An alternative method would be to constrain decision variables to integers. You would then be doing integer programming, not linear programming. Solver can handle it, but you should be aware that the solution methods are completely different and beyond the scope of the course. Note how different the answer is!

Integer Programming

<table>
<thead>
<tr>
<th></th>
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<th>Tables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carpentry</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>Finishing</td>
<td>15</td>
<td>10</td>
</tr>
<tr>
<td>Contribution</td>
<td>800</td>
<td>500</td>
</tr>
<tr>
<td>Value</td>
<td>4.00</td>
<td>2</td>
</tr>
</tbody>
</table>
d) Linear Program, where each cabinet now requires 12 hours

<table>
<thead>
<tr>
<th></th>
<th>Cabinets</th>
<th>Tables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carpentry</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>Finishing</td>
<td>12</td>
<td>10</td>
</tr>
<tr>
<td>Contribution</td>
<td>800</td>
<td>500</td>
</tr>
<tr>
<td>Value</td>
<td>5.00</td>
<td>2</td>
</tr>
</tbody>
</table>

The optimal solution changes because the shadow price on the finishing was positive; equivalently, we might say that the finishing constraint was binding. At the new optimum, both constraints are binding. Interestingly, even though the price of finishing for cabinets went down, the new optimum has us build more tables! We build more tables because cabinets use more carpentry than tables. When capacity is limited, then on the margin, tables become a better choice than cabinets.

Part C: Choice Problems

1. Select problem 11, 13, or 14 in Chapter 3 of *The Science of Decision Making* (pages 90-92). You do not need to solve the problem you select. Instead, read the problem and then answer questions (a) through (f) below.

11. (Subway Cars) The Transit Authority must repair 100 subway cars per month, and it must refurnish 50 subway cars per month. Each task can be done in its own facility. Each task can also be contracted to private shops but at a higher cost. Private contracting increases the cost by $2000 per car repaired and $2500 per car refurnished.

The Transit Authority repairs and refurnishes subway cars in four shops. Repairing each car consumes $1/150^{th}$ of the monthly capacity of its Evaluation shop, $1/60^{th}$ of the monthly capacity of its Assembly shop, none of the monthly capacity of its Paint shop, and $1/60^{th}$ of the monthly capacity of its Machine shop. Refurnishing each car consumes $1/100^{th}$ of the monthly capacity of its Evaluation shop, $1/120^{th}$ of the capacity of its Assembly shop, $1/40^{th}$ of the capacity of its Paint shop, and none the monthly capacity of its Machine shop.

(a) Write down letters to represent the decision variables in the problem.
Let $P =$ Vehicles Repaired by MTA; Let $F =$ Vehicles Refurbished by MTA

(b) Write one sentence describing what the decision maker wants to do (the objective).
The MTA wants to minimize the cost paid out to private contractors by choosing the optimal number of cars to repair and refurnish in house.

(c) Now, write out the objective function.
$Z = 2000(100 – P) + 2500(50 – F)$
(d) Write one sentence describing each constraint.

Evaluation Shop capacity (=1) is consumed at a rate of 1/150 per repair, 1/100 per refurb.

Assembly Shop capacity (=1) is consumed at a rate of 1/60 per repair, 1/120 per refurb.

Paint Shop capacity (=1) is consumed at a rate 1/40 per refurb.

Machine Shop capacity (=1) is consumed at a rate of 1/60 per repair.

(e) Now, write out equations that represent each constraint.

\[
\frac{1}{150} P + \frac{1}{100} F \leq 1
\]

\[
\frac{1}{60} P + \frac{1}{120} F \leq 1
\]

\[
\frac{1}{40} F \leq 1
\]

\[
\frac{1}{60} P \leq 1
\]

\[
P \geq 0
\]

\[
F \geq 0
\]
13. **(A Farmer)** A 1200-acre farm includes a well that has a capacity of 2000 acre-feet of water per year. (One acre-foot is one acre covered to a depth of one foot). The farm can be used to raise wheat, alfalfa, and beef. Wheat can be sold at $550 per ton and beef at $1300 per ton. Alfalfa can be bought or sold at the market price of $220 per ton. Each ton of wheat that the farmer produces requires one acre of land, $50 of labor, and 1.5 acre-feet of water. Each ton of alfalfa that she produces requires 1/3 acre of land, $40 of labor, and 0.6 acre-feet of water. Each ton of beef she produces requires 0.8 acres of land, $50 of labor, 2 acre-feet of water, and 2.5 tons of alfalfa. She can neither buy nor sell water. She wishes to operate her farm in a way that maximizes its annual profit.

(a) Write down letters to represent the decision variables in the problem.
Let W = wheat, AR = alfalfa grown, B = beef, and AS = alfalfa sold.

(b) Write one sentence describing what the decision maker wants to do (the objective).
The farmer wants to choose the amount of wheat, beef, and alfalfa to grow and the amount of alfalfa to buy so as to maximize profit.

(c) Now, write out the objective function.
\[ Z = 500W - 40AR + 1250B + 220AS \]

(d) Write one sentence describing each constraint.
Up to 1200 available acres are consumed at a rate of 1 acre per unit of wheat, 1/3 acre per unit of alfalfa, and 0.8 per unit of beef.

Up to 200 available acre-feet of water are consumed at a rate of 1.5 acre per unit of wheat, 0.6 acre per unit of alfalfa, and 2 per unit of beef.

The amount of alfalfa grown must equal the amount consumed in beef production plus the amount sold in the market. Note that, the amount sold in market can be negative (representing the purchase of alfalfa)

The amount of wheat, alfalfa grown, and beef cannot be negative.

(e) Now, write out equations that represent each constraint.
\[ W + \frac{1}{3}AR + 0.8B \leq 1200 \]

\[ 1.5W + 0.6AR + 2B \leq 2000 \]

\[ AS - AR + 2.5B = 0 \]

\[ W, AR, B \geq 0 \]
14. (Pollution Control) A company makes two products in a single plant. It runs this plant for 100 hours each week. Each unit of product A that the company produces consumes two hours of plant capacity, earns the company a contribution of $1000, and causes, as an undesirable side effect, the emission of 4 ounces of particulates. Each unit of product B that the company produces consumes one hour of plant capacity, earns the company a contribution of $2000, and causes, as undesirable side effects, the emission of 3 ounces of particulates and 1 ounce of chemicals. The EPA (Environmental Protection Agency) requires the company to limit particulate emission to at most 240 ounces per week and chemical emissions to at most 60 ounces per week.

(a) Write down letters to represent the decision variables in the problem.
Let A = product A, and let B = Product B

(b) Write one sentence describing what the decision maker wants to do (the objective).
The company wants to maximize firm profit; that is, the total contribution from products A and B.

(c) Now, write out the objective function.
Z = 1000A + 2000B

(d) Write one sentence describing each constraint.
The 100 weekly hours of plant capacity is consumed at a rate of 2 hours per unit of A and 1 hour per unit of B

240 weekly ounces of allowable particulate emissions are consumed at 4 ounces per unit of A and 3 ounces per unit of B.

60 weekly ounces of allowable chemical emissions are consumed at 1 ounce per unit of B.

The company can’t produce negative amounts of A or B.

(e) Now, write out equations that represent each constraint.
2A + 1B <= 100

4A + 3B <= 240

B <= 60

A, B >= 0
Using the SUMPRODUCT( ) Function in Microsoft Excel

SUMPRODUCT, as its name suggests, performs two functions on data organized in arrays (or vectors).

A vector is a mathematical object. You can think of it as a container designed to hold a list of values. Each value is an element of the vector.

Example: Vector A = \( \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} \) Vector B = \( \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ b_n \end{pmatrix} \)

SUMPRODUCT( ) multiplies the \( i \)th element of each array together and then sums all the products. In other words, SUMPRODUCT ( ) calculates the dot product of A’B

\[ = a_1b_1 + a_2b_2 + \ldots + a_nb_n = \sum_{i=1}^{n} a_ib_i \]

To use SUMPRODUCT ( )

1. Select the cell where you want your output displayed.
2. Type =SUMPRODUCT(
3. Do not hit return key.
4. Use the mouse to highlight the cells representing Vector A.
5. Hit the comma key.
6. Use the mouse to highlight the cells representing Vector B.
7. Hit the enter key.

The SUMPRODUCT ( ) value will be displayed in the cell.

SUMPRODUCT() might be used to calculate the present value of a income stream or annuity, to manage a portfolio of stocks, to keep track of a inventory, or to prepare an invoice.