Surface Simplification using Quadric Error Metrics
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Introduction
• Surface simplification reduces the number of triangles in a mesh while preserving important features.
• A multiresolution model provides variable levels of simplification.
• This algorithm provides a general method for efficient simplification of large triangle meshes.

Surface Simplification

Previous Work
• Vertex Decimation (Schroeder et al)
• Vertex Clustering (Rossignac and Borrel)
• Iterative Edge Contraction (Hoppe, Ronfard and Rossignac, Gueziec)
• The Quadric Error Metric (QEM) algorithm is an efficient method for contracting vertex pairs.
• QEM can contract non-edge vertex pairs.

QEM Algorithm
• The fundamental operations are vertex pair contraction.
• And the inverse, vertex pair splitting.

QEM Algorithm
• The Quadric Error Metric has a dual use.
• In the preprocessing stage the metric is used to compute a contraction cost for each valid vertex pair.
• We can then queue pairs keyed on cost.
• As we iteratively remove pairs of least cost the metric is used to compute the optimal replacement vertices.
Optimal Vertex Contraction

- A single optimum vertex is computed to replace a vertex pair.
- This vertex is optimal in a least squares sense.
- The triangles surrounding each of the original vertices are used to compute the error metric.

Optimal Vertex Pair Contraction in 2D

V* is at the center of ellipses of constant error

Quadrics

- Quadrics are second degree algebraic surfaces in 3-D.
- Ellipsoids, hyperboloids, paraboloids, cones, cylinders, etc.
- Described in homogeneous coordinates with symmetric 4x4 matrix Q.
- $v^TQv = 0$, $v^T = [x\ y\ z\ 1]$

Plane Equation

- Point normal representation of a plane:
  \[ n^Tv + d = 0 \]
  \[ n \text{ is the unit normal} \]
  \[ d \text{ is an offset from the origin.} \]
  \[ n^T = [a\ b\ c], a^2+b^2+c^2 = 1, v^T = [x\ y\ z] \]
- The plane equation:
  \[ ax + by + cz + d = 0 \]
Plane-Based Error Metric

- Squared distance of a vertex $v$ from a single plane:
  \[ D^2(v) = (n^T v + d)^2 = (ax + by + cz + d)^2 \]

- Error at a vertex based on multiple planes:
  \[ E_{\text{plane}}(v) = \sum_i D_i^2(v) = \sum_i (n_i^T v + d_i)^2 \]
  This can be written more compactly.

Homogeneous Coordinates

- Rewrite the equation for $D^2$ in homogeneous coordinates.
- Let $\mathbf{p}^T = [a \ b \ c \ d]$, $\mathbf{v}^T = [x \ y \ z \ 1]$ then
  \[ D^2(v) = (p^T v)^2 = (v^T p)(p^T v) = (v^T \mathbf{p} \mathbf{p}^T v) = (v^T \mathbf{K}_p v) \]

One $K_p$ Matrix per Triangle

- Each $K_p$ contains the coefficients of the plane equation in the outer product of $p$.
  \[ K_p = \mathbf{p} \mathbf{p}^T \begin{bmatrix} a^2 & ab & ac & ad \\ ba & b^2 & bc & bd \\ ca & cb & c^2 & cd \\ da & db & dc & d^2 \end{bmatrix} \]

Quadric Surface Equation

- For $\mathbf{v}^T = [x \ y \ z \ 1]$
  \[ v^T K_p v = a^2 x^2 + b^2 y^2 + c^2 z^2 + 2abxy + 2acxz + 2bcyz + 2adx + 2bdy + 2cdz + d^2 = \varepsilon \]
  This is the equation of a quadric surface, an ellipsoid.

Q is a Sum Over Local Triangles

- Each vertex in a mesh is the intersection of a set of planes defined by the triangles that meet at the vertex.
- The error associated with each vertex, weighted by the triangle area $w_p$ is:
  \[ \Delta(v) = \sum_p v^T w_p K_p v = v^T \sum_p w_p K_p v = 0 \]
  \[ Q = \sum_p w_p K_p \text{ is calculated for each vertex.} \]
  Q is symmetric so there are 10 floats to store for each vertex.

Local Triangles

- The $Q$ matrix at a vertex is the weighted sum of the $K_p$ matrices for the local triangles.
- When two $Q$ matrices are added during edge collapse, shared triangles are added twice.
**Contraction Cost**

- For a pair of vertices \( v_1 \) and \( v_2 \) find the optimal vertex placement, \( v^* \).
- The least squared error is from
  \[
  V(v^*^T(Q_1+Q_2) \, v^*) = 0
  \]
- We can solve for this for \( v^* \) as follows:

\[
\begin{bmatrix}
q_{00} & q_{01} & q_{02} & q_{03} \\
q_{10} & q_{11} & q_{12} & q_{13} \\
q_{20} & q_{21} & q_{22} & q_{23} \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

* The resulting error \( v^*^T(Q_1+Q_2) \, v^* \) is the contraction cost.

**Geometric Interpretation**

- The cost function \( v^*^T(Q_1+Q_2) \, v^* = 0 \) is an ellipsoid, an isosurface of constant error.
- The interior of the ellipsoid represents all points with error < \( \varepsilon \).
- The principle axes of the ellipsoid are in the direction of maximum curvature, minimum curvature, and the surface normal (approximately). These are defined by the eigenvalues and eigenvectors of \( Q_1+Q_2 \).

**Cost Function Degeneracy**

- The cost function \( v^*^T(Q_1+Q_2) \, v^* = 0 \) is degenerate when \( Q_1+Q_2 \) is singular.
- With one zero eigenvalue, infinite cylinders. Planes all parallel to a line.
- Two zero eigenvalues, parallel planes. Flat surface.
- In practice compute the condition number of \( Q_1+Q_2 \) to test for an optimal solution.
- Fall back to edge collapse.

**Future Direction/Work**

- Global error
- Aggregation
- Mesh Inversion
- Integrate QEM with PM: QPM